Nested cubic roots.
https://www.linkedin.com/feed/update/urn:li:activity:6463695029771870208

Let $a_n := \sqrt[3]{1+\sqrt[3]{2+\sqrt[3]{3+\sqrt[3]{4+\ldots+\sqrt[3]{n}}}}}\ , n \in \mathbb{N}$.

Prove that:

(**1**) $a_{n+1}^3 < 1+\sqrt[3]{2}\cdot a_n$ for any $n \in \mathbb{N}$;

(**2**) Sequence $(a_n)_{\mathbb{N}}$ is convergent.

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1. Noting that $k \le 2^{3^{k-2}}(k-1)$ for any $k \in \mathbb{N}\setminus\{1\}$ (equality holds only if $k=2$)

we obtain $a_{n+1}^3 = 1+\sqrt[3]{2+\sqrt[3]{3+\sqrt[3]{4+\ldots+\sqrt[3]{n+\sqrt[3]{n+1}}}}} <$

$$1+\sqrt[3]{2+\sqrt[3]{2^{3^{3-2}}\cdot 2+\sqrt[3]{2^{3^{4-2}}\cdot 3+\ldots+\sqrt[3]{2^{3^{n-2}}(n-1)+\sqrt[3]{2^{3^{n-1}}\cdot n}}}}} =$$

$$1+\sqrt[3]{2+\sqrt[3]{2^{3^{3-2}}\cdot 2+\sqrt[3]{2^{3^{4-2}}\cdot 3+\ldots+\sqrt[3]{2^{3^{n-2}}(n-1)+2^{3^{n-2}}\sqrt[3]{n}}}}} =$$

$$1+\sqrt[3]{2+\sqrt[3]{2^{3^{3-2}}\cdot 2+\sqrt[3]{2^{3^{4-2}}\cdot 3+\ldots+2^{3^{n-3}}\sqrt[3]{(n-1)+\sqrt[3]{n}}}}} =\ldots=$$

$$1+\sqrt[3]{2+2\sqrt[3]{2+\sqrt[3]{3+\ldots+\sqrt[3]{(n-1)+\sqrt[3]{n}}}}} = 1+\sqrt[3]{2}\cdot a_n.$$

2. First we will prove that $a_n < \sqrt[3]{4}$ for any $n \in \mathbb{N}$.

Indeed, $a_1 = 1 < \sqrt[3]{4}$ and $a_2 = \sqrt[3]{1+\sqrt[3]{2}} < \sqrt[3]{4} \Leftrightarrow \sqrt[3]{2} < 3$.

For any $n \in \mathbb{N}$ assuming $a_n < \sqrt[3]{4}$ we obtain

$a_{n+1}^3 < 1+\sqrt[3]{2}\cdot a_n < 1+\sqrt[3]{2}\cdot\sqrt[3]{4} = 3$ and, therefore, $a_{n+1} < \sqrt[3]{3} < \sqrt[3]{4}$.

Thus, by Math Induction, $a_n < \sqrt[3]{4}$ for any $n \in \mathbb{N}$ and since $a_{n+1} > a_n$ for any $n \in \mathbb{N}$ we can conclude that sequence $(a_n)_{\mathbb{N}}$ is convergent as increasing and bounded from above.